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On the Value Principles Guiding the Development of Geometrical Axiomatic Systems

The 19th century saw the rise of non-Euclidean geometries, developed in the work of Bolyai, Lobachevsky, Riemann, Gauss, and Beltrami, among others. Beyond constituting new insightful mathematical results, the formalization of geometric models that seem to contradict our everyday Euclidean intuition inevitably poses major epistemological questions. In particular, we are concerned with the considerations, implicit and explicit, that guide the choice of axioms when setting up a particular geometric theory. The first description of a non-Euclidean geometry, namely Bolyai and Lobachevsky's introduction of hyperbolic geometry, constitutes a radical turning point: the realization that other geometries are possible. But the moment we accept that Euclidean geometry is not the only plausible geometric system, a natural question follows: how do we decide what constitutes an "valuable" geometric model worth studying and what doesn't? From a purely 20th century axiomatic point of view, it would appear that such a choice can essentially be made arbitrarily, as long as the proposed set of axioms are consistent with one another. However, we argue that this seemingly purely-logical standpoint actually rests on a long history of shifting criteria for deciding what grants value to a mathematical theory. In this essay, we will survey the major developments of non-Euclidean geometries and infer from them the principles that guided their corresponding choice of axioms.

With the publication of Euclid's *Elements* in 300 BC*,* Euclidean geometry establish itself as the standard geometrical theory. Two key aspects of the *Elements* secured its success: first and foremost, the concepts and theorems that it presents (in the form of definitions, postulates, common notions, and propositions) appear to be *intuitive* and *evident*. In order words, they match our geometrical perception of physical space.[1] This follows from the fact that Euclid pursued a *non-algebraic* treatment of geometry; namely, Euclidean geometry is guided by the assumption that the goal of a geometric theory is to model physical space (which is the *unique subject matter* of the theory), and thus consists of the study of propositions that are true of physical space. Second, Euclid's treatment of geometry was *rigorous* and *formal,* establishing its basic grounding fundamental concepts and then proceeding purely with logical reasoning.

However, as centuries passed, the demand for logical correctness started to overweight the need for a faithful representation of the subject matter. For years, mathematicians attempted to prove Euclid's 5th postulate from the other four to no avail. In the 19th century, Bolyai and Lobachevsky's (independent) investigations, motivated by these unsuccessful attempts, took a completely different approach: instead of trying to *prove* Euclid's 5th postulate, they replaced it with a new axiom and thus constructed a new geometry, which was later called *hyperbolic geometry.* That is, Euclidean and hyperbolic geometry agree on all their axioms except one, Postulate V. Importantly, they both showed that hyperbolic geometry can realize Euclidean geometry, and thus that hyperbolic geometry is consistent relative to Euclidean geometry. Given the predominant non-algebraicist view of geometry at the time, this relative consistency was critical in justifying the validity of these new non-Euclidean theories.

Moreover, we also infer that *generality* is a value principle guiding their axiomatic systems, given that Bolyai's so-called *absolute geometry* constitutes the common ground for both Euclidean and hyperbolic geometry, and the proof of relative consistency was

[1] For example, Postulate 1 says that given any two points such as A and B, there is a line AB which has them as endpoints. This is a "self-evident" statement which seems inherently true to us.

motivated by a desire of absorbing Euclidean geometry into a more general system (Torretti, p. 62). We remark that while Bolyai and Lobachevksy took a clear step into the analytical (as opposed to intuitive) study of space, their primary research goal was still the study of physical space, and the two mathematicians were not trying to set forth a formalist view of mathematics.

This claim is supported by the following two observations. First, Bolyai and Lobachevksy never put into question any of the other axioms present in Euclid's model, which could have been falsified in a similar way. Thus, their motivation behind falsifying Euclid's 5th postulate was not to test "groundless" axioms, as if any of them could have been negated in the way that Hilbert later did. Second, they both argue that *either* the system of geometry based on the hypothesis that Euclid's Postulate V is true, *or* the system based on the opposite hypothesis is. Crucially, they are implying that there *exists* an objective answer to this question, even if they are unable to answer it. As Bolyai argues, "it rests undecided whether [hyperbolic geometry] or [Euclidean geometry] is true in *actual fact*" (Bolyai, p.108). This is also why Bolyai remarks that some of his theorems are "valid independent of whether [hyperbolic geometry] or [Euclidean geometry] is *true in reality*" (Bolyai, p.85). They both believed that this question could be answered by means of a physical measurement, and in fact Lobachevsky attempted to settle this question by evaluating astronomical data himself, which turned out to not be precise enough (Lobachevsky, p.49). Therefore, they believed that the question of *which geometry is true* is a factual one, which implies that the discipline of geometry has a unique subject matter.

A similar approach was undertaken by Eugenio Beltrami a few years later. The value principles underlying his approach to geometry were the same as Bolyai and Lobachevsky's: the foremost requirement was to describe physical space, and the second one was to provide logical consistency. In fact, while Beltrami similarly believed that the truth of a geometrical system can be resolved through an empirical measurement (Beltrami, p. 58), he took a step towards a metamathematical approach to the different axiomatic systems. He showed that the relative consistency initially shown by Bolyai and Lobachevsky actually goes *both ways*: that is, hyperbolic geometry can interpret Euclidean geometry, but Euclidean geometry can also interpret hyperbolic geometry. In fact, Betrami proved this claim in a less axiomatic way than Bolyai and Lobachevsky, by showing how to realize hyperbolic geometry on a surface of constant negative curvature, called the *pseudosphere.* In this interpretation, lines are represented by geodesics on the pseudosphere. We again remark how important it was for the first proponents of non-Euclidean theories to define their terms *with respect to* Euclidean geometry in order to attach higher intuitive meaning to the, given that their primary value criteria for a meaningful geometric theory was the faithful representation of its subject matter. Beltrami in fact referred to the Euclidean model as a "real substrate", which is what motivated him to find a real substrate for hyperbolic geometry in a curved surface in Euclidean space (Beltrami, p. 7). This resonates with Bolyai statement that his (hyperbolic) L-lines "*play the role* of straight lines" (Bolyai, p. 88), implying that new geometric theories must always be understood *with respect to* Euclidean geometry, which is the one that matches our geometric intuition.

The equivalence between Euclidean and hyperbolic geometries was later picked up by Henri Poincaré, but who read this result in a completely different manner. Poincaré did *not* hold a non-algebraic view of geometry – in his regard, algebraic methods are based on the assumption that the objects of a mathematical theory are only relevant as insofar they sustain the relations we require them to have to one another, and this is all the mathematical theory requires (Torretti, p. 141). Because we can interpret all geometries through the lenses of one another, we can then translate between them equally, and thus there is no one geometry that actually conforms to physical space. More pictorially, he constructs a "sort of dictionary" that allows us to match "in one-to-one correspondence" the hyperbolic terms with the Euclidean ones (Poincaré, p. 37), which illustrates how both theories yield the same theorems about space and are mutually consistent.

While this equivalence had already been shown by Beltrami, Poincaré's key take-away is drastically different: unlike his predecessors, he rejects the claim that geometrical axioms are "experimental truths" and that geometry is an "experimental science" (Poincaré, p. 56), and instead claims that "Euclid's postulate could not be experimentally demonstrated" (Poincaré, p. 59). In fact, more strongly, Poincaré argues that the question of *is Euclidean geometry true?* does not "make any sense" (Poincaré, 43). It does not make sense because Poincaré does not attach a subject matter to geometrical theories, and therefore the main value guiding the choice of axiomatic system simply *cannot be* the truthfulness of the system. To him, the choice of geometry is then purely *conventional* – because it does not make sense to ask which geometry is true, we can just choose one arbitrarily. Poincaré only restrains this choice with one condition; namely, "it is only limited by the need to avoid all contradiction" (Poincaré, p. 43). This perspective takes a step closer to the axiomatic theories of the 20th century, as we will develop with Hilbert. However, Poincaré's conventionalism hides important value principles: for him, conventionalism is in turn guided by *usefulness*, *simplicity*, and *alignment with our intuitions.* In his view, these are the actual reasons for why "Euclidean geometry is now and will remain the most *useful* geometry" (Poincaré, p. 43).

A similarly strong algebraic view of geometry was held by Felix Klein, who was the first to pursue a group-theoretic approach to geometry. This is evidently drastically opposed to Bolyai, Lobachevsky, and Beltrami's conception of geometry. Because group theory is the canonical example of an algebraic theory (since a group does not have any specific subject matter attached to it), Klein makes the algebraic approach to geometry of the late 19th century as explicit as it can get. In Klein's view, a geometry is defined by a space and a *group of transformations* (Klein, p. 3), and is hence clearly detached from any subject matter. In fact, Klein states that we can "dispose with the concrete conception of space" (Klein, p. 4).

But from his theory we infer another key new value principle, which is that of *unification.* Namely, Klein showed that *projective geometry* can be understood as a *unifying frame* for the rest of geometry, by illustrating how they correspond to special cases of projective geometry. As part of the Erlangen program, Klein managed to characterize all geometries as subgroups of the projective group. Therefore, while both Poincaré and Hilbert were strong proponents of the algebraic view of geometry, their value criteria behind their choice of geometric system was very different: for Poincaré, there was no such thing as a geometry "above others" (they are all equivalent to him), unlike the hierarchical system proposed by Klein. It is also worth noting that Klein's approach showed that spherical, Euclidean, and hyperbolic geometry are all consistent if projective geometry is consistent.

Klein and Poincaré's approaches anticipated the modern axiomatic method developed by Hilbert in the 20th century. Hilbert shares the non-algebraic view of geometry of Klein and Poincaré, but the value principles that guide his choice of a particular axiomatic system is fundamentally different. Hilbert takes the final step towards an axiomatic perspective which is *completely* detached from the subject matter. That is, for Hilbert the one and only requirement for an axiomatic system to be valid is for it to be *consistent* (i.e., no two axioms can be contradictory), and there are no more principles guiding the choice of an axiomatic system. In Hilbert's view, not only is the axiomatic system detached from any subject matter, but the fundamental elements of the theory are plainly viewed as *placeholders,* and so we could replace the terms *point, line,* and *plane,* by *mug, chair,* and *apple.* For Hilbert, a concept is properly defined only by means of a formal axiomatization; once the axiom has been laid down, "it exists and it is true" (Frege-Hilbert correspondence, p. 39). Thus, there is no prior intuition or meaning of the different geometric concepts (such as *line*) – such meaning is *only* established by lying down the axioms. This is why Hilbert's proposed geometric system contains no definitions. Moreover, Hilbert constructs a characterization of the Euclidean geometry which is *complete*, as this was his main (and only) objective to fulfill.

It is relevant to note the following two nuances in Hilbert. First, his position (although similar) is *not* the same one as that of Poincaré. While Poincaré concludes that the choice of geometry is conventional, he is not implying that geometry maintains no relationship at all with the physical space. Poincaré concludes that the choice of geometry does not matter *only* because all geometries can be equally applied to the same physical world, albeit in different ways (i.e., they are re-interpretations of the same concept). Poincaré allows us to pick any equivalent interpretation of "straight line" because it has been mathematically shown that the different geometries are relatively consistent to one another. But Hilbert is completely stripping the concept of "straight line" of *any* meaning – the word is empty until an axiom is laid down, and such axiom can be chosen arbitrarily as long as it is consistent with the rest of axioms. Hence for Poincaré the nature of "solid bodies" around us matters for his value criteria, whereas for Hilbert it does not.

Second, we remark that not all logicians contemporary to Hilbert who shared the same axiomatic approach to mathematics agreed with Hilbert's conclusions. For example, Gottlob Frege, who famously engaged with Hilbert on a debate about the nature of mathematical axioms, believed that no proof of consistency was needed, given that semantic meaning of words exists prior to setting up the axioms, and thus "from the truth of axioms it follows that they do not contradict one another" (Frege-Hilbert correspondence, p. 39). Thus, for Frege, axioms can be either true or false, whereas for Hilbert falsehood is not possible. Importantly, Hilbert's quest for consistency was partially shattered when Kurt Gödel published his *incompleteness theorems.* Essentially, he showed that what Hilbert accomplished in geometry (i.e., a consistent geometric system) could *not* be achieved for arithmetic. Because a consistent system realizing arithmetic cannot exist, Hilbert's value criteria for establishing axiomatic systems appears to be perhaps unnecessary.

Moreover, it seems that Hilbert's axiomatic method leaves the quest for establishing value principles guiding mathematical theories at a dead end. Namely, one would infer from Hilbert's texts that any axiomatic system which is consistent is equally valid to constitute a full mathematical theory. However, this conclusion is disappointing: at the end of the day, only certain mathematical theories prevail, and mathematics research always implicitly follows some general consensus of *value* and *interest.* In this essay, we have provided a historical account on the different axiomatic systems for geometry which provides relief to this disappointing conclusion: namely, Hilbert's formal approach stands on centuries of shifting value criteria for what constitutes a worthwhile geometric theory.

First, we have seen a shift from a non-algebraic approach to geometry to a purely algebraic one. On the non-algebraic approach, Euclid begins by formalizing our conception of physical space as faithfully as possible. We then saw how Bolyai, Lobachevsky, and Beltrami shift the value system towards its axiomatic presentation (and departing from our innate Euclidean intuition) by falsifying Euclid's 5th postulate, which also brings forth the criteria of *generality*, while maintaining the possible *empirical realization* of the geometry as a key value principle. On the other hand, Poincaré and Klein depart to an algebraic view of geometry, in which a geometric model can no longer be chosen based on its empirical truth. However, Poincaré prioritizes the value of *conventionality* (which is in turn guided by *simplicity*, *usefulness,* and *agreement with the properties of natural solids*) over Klein's value of *unification.* Thus, we arrive at Hilbert's loosened principle of *completeness* only after having traversed a long history of shifting criteria of mathematical value.

We then conclude that the development of new geometrical theories and of new values guiding mathematical theories go hand in hand: new mathematics will induce new perspectives on the guiding values, which will in turn motivate the development of new mathematics. Moreover, while the disparity of value criteria across the different geometric theories that we have survey has been made apparent, we also conclude that they share one aspect in common. Namely, no theory has ever disregarded Euclidean geometry, and they are all related to it and described with respect to it in some way or another: Bolyai, Lobachevsky, and Beltrami consider the relative consistency of hyperbolic geometry with respect to the Euclidean one and define their terms as analogies of the Euclidean ones, Klein's unified approach needed to include Euclidean geometry under its umbrella, Poincaré claims that Euclidean geometry is and will remain the most useful one and emphasizes the translation equivalences between Euclidean and non-Euclidean geometries, and Hilbert decides to present his axiomatic method by proving the consistency of Euclidean geometry. Therefore, despite the significant changes in the principles that guide our choice of axiomatic system, Euclidean geometry has always persisted in the background as the canonical example, and we have never departed from it entirely.

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