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Kant on the Syntheticity of Mathematics: a Geometric Perspective

Mathematics, and in particular geometry, has played an essential role in many of the central epistemological theories in the history of philosophy. From Platonic forms to Kant's analytic and synthetic distinction, pure mathematics and the way in which human cognition is able to reason mathematically and acquire knowledge has long intrigued philosophers. Kant notoriously entitled one of the main chapters in his *Prolegomena* as the question: *How is pure mathematics possible?*. In this essay, I examine Kant's claim that mathematical judgements are synthetic a priori from the perspective of geometry, and propose that Kant's construction of the concept can instead be replaced by a new axiomatic perspective of how we come to a concept, which in turn yields a broader definition for Kant's notion of pure intuition of space. I argue that this change does not modify Kant's main claim on the syntheticity of mathematics, yet it allows for non-Euclidean geometrics to be included within the Kantian framework.

Two central concepts in Kant's argumentation are *analytic* and *synthetic* judgements. Analytic judgements are those which are "merely explicative and add nothing to the content of the cognition" (Kant, p. 16), because the predicate concept is already contained in the subject.[1] Therefore, analytic propositions are all *a priori* judgements; i.e., are independent from experience. This term is opposed to *a posteriori* judgements, which are those that require empirical evidence. On the other hand, synthetic judgements are those which are not true by definition because the predicate concept is *not* contained in the subject. Some synthetic judgements are *a posteriori* because their origin is empirical; for example, "The weather is cold." However, Kant's striking claim is that there exist synthetic a priori judgements, which he proves by claiming that *all mathematical judgements are synthetic a priori* (Kant, p. 18).

[1] For example, the proposition "All bachelors are unmarried" is analytic, because it does not augment our cognition of the concept of "bachelors", and rather only states a fact that is deducible by definitions and logic alone.

Let us review the relevant terminology that is involved in this claim. For Kant, *intuition* (which is contrasted with the notion of *concept*) is a form of representation which constitutes the starting point of any form of cognition. *Sensibility* is the faculty that allows us to have intuitions, which can either be *pure* (equivalently, *a priori)* or *empirical.* He argues that space (and time) are an *a priori form of intuition,* where the *form of intuition* refers to our only way of possessing pure intuitions. That is, space and time do not exist on their own in the physical external world,[2] but rather are subjective forms of our sensibility. Kant then exhibits the syntheticity of geometric propositions by noting that any geometric proposition is based on our a priori conception of space (Kant, p. 35).

I will now examine this argumentation more closely and point out two apparent weaknesses in Kant's argument: first, when Kant describes the notion of *construction of the concept,* he does not fully distinguish between the process of constructing axioms and definitions from that of building proofs and theorems. I will argue that one acquires the concept when axiomatizing it and defining it precisely, and that it is in this first step where the pure intuition of space is required.[3] This new understanding of *concept* removes the need for Kant's *construction of the concept,* since it is no longer necessary to exhibit the object in pure intuition. Second, I will illustrate how the removal of the construction of the concept allows for a looser (i.e., vaguer) definition of pure intuition of space. I argue that this provides a way of including non-Euclidean geometries in the Kantian framework, which is originally rooted in the conception that our *a priori* intuition of space is Euclidean and does not conceive the possibility that abstract geometric theories can be developed independently from the physical space. In making these changes I will conclude that none of them invalidates Kant's claim that mathematical judgements are synthetic a priori.

Firstly, Kant claims that the "essential feature of pure mathematical cognition […] is that is must throughout proceed not from concepts, but always through the construction of concepts" (Kant, p. 20), and that it is in the construction of a concept where nonempirical intuition is required. When he illustrates such a construction of the concept, he gives the example of proving that the sum of the angles in a triangle equals two right angles. He argues that if we give the concept of a triangle to a philosopher, "he can

[2] As Kant puts it, they are not a thing-in-itself (in German, *Ding an sich*).
[3] This is opposed to it being necessary at the second step where we build a proof and obtain a theorem from those axioms and definitions, which is a purely analytic process and does not fundamentally require any *a priori* intuitions.

analyze and clarify the concept of a straight line, or of an angle, […] but he cannot come upon any other properties". Philosophy, Kant says, is "discursive" and coming from mere concepts (Kant, p. 29). However, if we give the triangle to a geometer, he is able to *construct a triangle* and deduce that its angles sum up to two right angles. Kant says: "In this fashion, through a chain of inferences **guided throughout by intuition,** he arrives at a solution of the problem" (Kant II, A715-757/B732-745).

I claim that in this argumentation Kant is not faithfully describing how the mathematical process works because he is not distinguishing between two crucial steps in any mathematical reasoning: to prove any geometric proposition, we first need to provide a set of axioms (e.g., Euclid's postulates) and a set of definitions, and then *from these fundamental notions* we are able to derive a set of theorems, which follow *purely logically* (i.e., analytically) from the fundamental notions, and not "guided throughout by intuition", as Kant claims.[4] It is thus in providing the definition of a straight line and a point, for example, where our a priori intuition of space is required.[5] Let us consider Euclid's *Elements:* once the common notions, postulates, and definitions have been properly set up, then any of the propositions in Euclid's book follows by logical deduction alone (and, in particular, so does the proposition that the sum of the angles of the triangle is equal to two right angles[6]). In fact, we could feed the set of fundamental notions into an automatic software verification tool, and it would be able to prove those propositions.[7]

However, we would not say that a computer has any a priori intuition of space, which is a property of human cognition. Hence, if there is any point of the mathematical

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[4] Kant also mixes up axioms and theorems when citing Euclid, further supporting the claim that he is carelessly gluing these two very distinct concepts together. As pointed out by Friedman, "at A25 he calls the proposition that two sides of a triangle together exceed the third a fundamental proposition (*Grundsatz*). This latter is of course not an axiom in Euclid, but a basic (and therefore fundamental) theorem (Prop. 1.20)." (Friedman, p. 83).

[5] Riemann would agree with this argumentation, since he also points out that the importance of understanding how we set up the postulates and definitions in geometry has been overlooked: "Geometry takes for granted the notion of space as well as the fundamental first principles used in constructions carried out in space. Only nominal definitions are given of these basic concepts, while the essential role in determining their properties is played by the axioms. The relationships between the assumptions embodied in these axioms, however, remain obscure. It is not clear whether, and if so to what extent, they are necessarily linked; or whether, *a priori*, they are even possible. This obscurity has existed from Euclid to Legendre, to name the most famous of recent geometers, but neither the mathematicians nor the philosophers who have concerned themselves with this problem have dispelled it". (Riemann, p. 257).

[6] More precisely, this proposition follows from Postulate II, Common Notion IV, Playfair's axiom, and Proposition 29.

[7] This has in fact been done: https://hal.archives-ouvertes.fr/hal-01612807/file/ProofCheckingEuclid.pdf. As the authors point out in the abstract, such automatic proving required using a language "closely related to that used in Tarski's formal geometry", and they also needed to fill Euclid's logical gaps and correct his errors.

process in which the a priori intuition of space is required, then that point is the process of writing down the fundamental notions. Moreover, this view is supported by the following two mathematical observations. First, definitions play a crucial role in mathematics and go well beyond constituting a purely descriptive statement – in many cases, it is precisely in defining the concepts where the bulk of the reasoning occurs. A notorious example is Stokes' theorem, which required many years to be proven, but once the necessary terms were defined, the proof became trivial. As Michael Spivak states in his book *Calculus on Manifolds,* "Stokes's theorem shares three important attributes with many fully evolved major theorems: 1. It is trivial. 2. It is trivial **because the terms appearing in it have been properly defined.** 3. It has significant consequences. Since this entire chapter was little more than a series of **definitions which made the statement and proof of Stokes' theorem possible,** the reader should be willing to grant the first two of these attributes to Stokes' theorem." (Spivak, p. 104). Another example is the rise of non-Euclidean geometries and the long historical back-and-forth about the validity of Euclid's Fifth Postulate also demonstrates that the key contribution of Bolyai and Lobachevsky, among others, was in *setting up an alternative set of axioms,* rather than in the theorems that follow those.

Second, gap-free mathematical logical systems are indeed possible. While in the mathematical context of Kant, Euclid's theorems did not actually follow purely logically from the axioms given that there were certain lacunae,[8] Tarski provided a system of foundations for Euclidean geometry that fixed them. This is, in particular, why Tarski's axiomatization (and not Euclid's) is required for computer proof verification. The fact that gap-free systems *exist* strongly supports the argument that a concept is acquired by means of axiomatizing it, and after that logic follows alone.

From this discussion, we reach the following conclusions. First, my newly proposed definition of how we acquire concepts (namely, by axiomatizing and defining them precisely) removes the need for Kant's construction of the concept. Kant argues that we need the construction of the concept in a mathematical proof in order to make it valid. However, by understanding the notion of the concept as *already* axiomatized, it follows that it is no longer necessary to exhibit the object in pure intuition while proving mathematical statements about it. Second, this change does not contradict Kant's claim that mathematical judgements are synthetic a priori. Even if theorems can be deduced

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[8] One such example is Book I Proposition 1 of Euclid's *Elements,* which requires continuity arguments that were not known prior to Cauchy and Dedekind in the 19th century.

analytically from the set of axioms and definitions, the process of acquiring the concept (i.e., to axiomatize it) is still synthetic. More precisely, there exists a collection of intuitive judgements which allow us to come to a definition[9] in a synthetic manner. Therefore, by transitivity it follows that the theorems must be synthetic too, because their necessity is inevitably based on those fundamental synthetic principles.

Importantly, we see that our ability to acquire concepts by defining them precisely crucially rests on some existing prior form of intuition. From this fact we deduce that all humans have some vague pre-conceived idea of space, which is rather non-specific. This *must* be a vague notion, because formalizing it amounts precisely to getting to the concept, but this intuition precedes (and in fact, makes possible) such a formalization. In order to get to *the* concept of space, as I defined it in the first part of this essay, we need to fully give the axioms and definitions that describe it. It is in this second step where we can decide whether to give, for example, the Euclidean or the hyperbolic axiomatization of space. But either way, both axiomatizations *first* require that our cognition is able to reason about fundamental notions about space, such as magnitude, lines, and points, which is where our pure intuition of space resides. This means that Kant's notion of *pure intuition of space* does not amount to having a precise Euclidean axiomatization of it, because the intuition precedes the concept. Thus, it must correspond to (what I call) a *pre-theoretic notion* of space in broad and under-specified terms.

One main advantage of this proposed re-formulation of the Kantian framework is that it allows for non-Euclidean geometries to be understood within it. This is a necessary amendment given that Kant's account of the construction of the concept is evidently grounded on Euclidean geometry, and in particular on the assumption that our pure intuition of space is Euclidean. This is already apparent in the previous description of what Kant considers to be the exemplary construction of the concept, given that it rests on the Euclidean account of the geometry of a triangle.[10] Kant does not even conceive the

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[9] Remark: we make no distinction between definitions, axioms, and postulates. For the purposes of this essay, we view all three as different types of "fundamental notions", and we are not concerned with their differences. In fact, making such a distinction requires a lengthy discussion, as it is evident from the famous Hilbert vs Frege debate in the 20th century on the difference between axioms and definitions when establishing an axiomatic system. However, my stance in this essay is very similar to Hilbert's argument: If the only synthetic part of mathematics is axiomatization, and the activity of axiomatization is essentially the same as that of definition, then it seems like the usefulness of the distinction is undermined. Hilbert reaches the similar conclusion that such a distinction is not needed, and that in fact we define concepts by axiomatizing them.

[10] This is because in hyperbolic and spherical geometry, for example, the sum of the angles in a triangle is not necessarily equal to two right angles.

possibility that abstract geometric theories can be developed *independent* of physical space – namely, he makes no distinction between pure and applied geometry.

Therefore, by changing the definition of how we get to the concept and hence removing Kant's construction of the concept and loosening Kant's description of pure intuition of space, it should not necessarily follow that space is Euclidean, and in particular it then admits any alternative axiomatization. Again, because the concept is only achieved when axiomatized, the pre-theoretic notion of space cannot belong a priori to any particular axiomatic system, given that this intuition is a necessary prior condition for getting to the concept. Namely, to represent the world to ourselves, we do not need to include a full specification of a geometric theory. However, all of the possible axiomatic systems still yield synthetic a priori judgements, because the syntheticity of such judgements originates in our broad pre-theoretic notions of space that allow us to reason about geometry in the first place, and which *do not* necessarily fall into any particular axiomatization theory.

In conclusion, Kant's claim that mathematical judgements are a synthetic a priori constitutes a stepping stone in the history of metaphysics. The argumentation behind this claim crucially relies on his proposed terms of *construction of the concept* and of *pure intuition of space.* In this essay I have provided evidence that the definition of both concepts should be revised: on the one hand, the construction of the concept fails to distinguish between synthetically providing axioms and definitions for a mathematical theory from analytically deducing theorems from them. On the other hand, his notion of pure intuition of space does not allow for non-Euclidean geometries to fit within the Kantian framework. By introducing a new definition of what it means to acquire a concept (i.e., to axiomatize it) and thus loosening the a priori intuition of space into a vaguer pre-theoretic notion resolves both problems. Moreover, this amendment does not jeopardize Kant's central claim about the syntheticity of mathematics.

Works Cited

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